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Critical Slowing Down on the Dynamics of a Bistable Reaction-Diffusion System in the Neighborhood of Its Critical Point

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We investigate the nature of some critical decay processes in a bounded bistable reaction-diffusion system, through a perturbative expansion of its nonequilibrium potential. We elucidate the scaling behavior of the damped relaxation time.

KEY WORDS:

I. INTRODUCTION

Pattern formation and propagation is one of the main concerns in the study of complex systems and has became a very active field of research in physics, chemistry and biology, both from the experimental and from the theoretical points of view. The description in terms of reaction-diffusion (RD) schemes provides a fruitful source of tractable models. Some of those models have been extensively investigated during the past years.[1–6] Nevertheless there are still questions to be answered. We will address here the one corresponding to the expected critical slowing down (CSD) of the evolution of the patterns, when a bistable RD system approaches its critical point.

The specific system we shall focus on, is the same one for which we have recently analyzed the structural stability of the stationary patterns.[7]

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That is the so-called “hot-spot” model (a superconductive resistor immersed in a heat bath) whose formulation, already scaled, reads

$$\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T}{\partial x^2} - T + Y[T-\theta] \tag{1}$$

Here $T(x,t)$ is a real field representing the distribution of temperature along the resistor for $-L < x < L$, Y is the Heaviside step function, θ is the scaled critical temperature of the superconducting transition and the temperature of the heat bath has been taken as zero.[6–11] We shall apply Dirichlet boundary conditions (DBC) by fixing the values of T at the boundaries:

$$T(-L,t) = T(L,t) = 0 \tag{2}$$

For bistable dynamics, the resistor has superconducting phases at $T < \theta$ and a normal phase for $T > \theta$ coexisting in a stationary state.[10, 11] The piecewise linear form of the dissipation term allows an analytical treatment. We recall that most of the qualitative results obtained for this piecewise linear model could be straightforwardly extended to systems with smoother bistable potentials.[7, 12, 13]

Equation (1) has a critical point, which separates a bistable dynamics from a monostable one. That kind of critical point, where two fixed points of different stability collapse, were already analyzed within a Lyapunov functional approach for monostable RD systems in refs. 14 and 15, and for bistable RD systems in refs. 16 and 17. As it was indicated in refs. 14 and 16, a critical slowing down process in the evolution of the spatial structures is expected in the neighborhood of the critical point. A theoretical and numerical analysis of those phenomena, based on a Landau expansion of the nonequilibrium potential, can be found in ref. 17 where a mechanical analogy is also discused.

In this communication we will see how the problem of pattern formation and relaxation can be analyzed in the critical region in terms of a single scalar parameter. That parameter measures the distance from the critical point, along one of the axis of the parameter space. Our analysis will be carried out through an asymptotic time-expansion of the nonequilibrium potential of the system.

II. PATTERN FORMATION

Let us briefly review, for completeness, the symmetric stationary patterns of Eq. (1) and their stability.[11] Our control parameters are going to be L and θ.

A. Stationary Patterns

Equation (1) is solved in the stationary case by proposing different linear combinations of hyperbolic functions for $T(x)$, depending on whether $T > \theta$ or $T \leqslant \theta$. Those functions, as well as their first derivatives, are matched at the interphase, namely x_c, where $T(x_c) = \theta$. From that matching we get

$$x_c = (L/2) \pm \operatorname{argcosh}\left(\frac{1-2\theta}{1-2\theta_c}\right) \tag{3}$$

being $\theta_c = \frac{1}{2}(1 - 1/\cosh(L))$. θ_c approaches asymptotically to 1/2, thus for $L \gtrsim 7.5$ the size effect is no longer relevant to it. By imposing the announced DBC, we get the particular solutions of the stationary case. The resulting structures have a hot (activated) region at $-x_c < x < x_c$, where $T > \theta$ and Joule-dissipation occurs. In Fig. 1a we show x_c/L as a function of θ for several values of L. The upper (lower) branch in Fig. 1a corresponds to the upper (lower) sign in ec. (3). The coalescence of both non-uniform structures takes place when θ reaches the critical value θ_c. At that critical point, the dynamics of the system changes from bistable ($\theta < \theta_c$) to monostable ($\theta \geqslant \theta_c$). The solution $T = 0$ becomes the only locally stable attractor and the two nonuniform structures collapse.

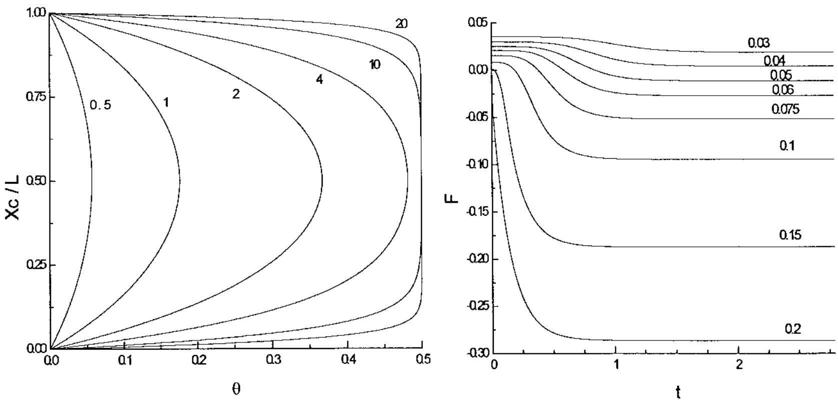

Fig. 1. Description of the system. Left) 1a: Stationary patterns: x_c/L vs. θ. The curves are parameterized with the values of L. The existence of a maximum allowed value of θ for each L can be appreciated; moreover, the solutions are twofold. Right) 1b: time behavior of the nonequilibrium potential F during decay processes for different values of $\Delta\theta$. Note that for $\Delta\theta \to 0$ the system becomes critically damped. The curves are labeled with their $\Delta\theta$ values and we fixed $L = 1$.

In the vicinity of the critical point, Eq. (3) takes the asymptotic form:

$$x_c - L/2 = \pm\sqrt{\Delta\theta} \tag{4}$$

where we have introduced $\Delta\theta = \theta_c - \theta$.

B. Local Stability

The standard linear stability analysis for the resulting stationary structures, shows that the upper branches in Fig. 1a correspond to locally stable patterns, the lower ones being unstable. For each parameter set, the structure with the largest hot region (i.e., bigger x_c and then bigger Joule-effect dissipation) is stable, whereas the other one is unstable. Those patterns occur beyond the linear regime, where the minimum entropy production principle is no longer valid. The locus of the dividing points between both branches determines the line of marginal stability. The stationary homogeneous solution $T = 0$ is locally stable for any values of the parameters.[11]

C. Global Stability

For Eq. (1), the nonequilibrium potential functional $F\{T\}$ takes the form:

$$F = \int_{-L}^{L} \left[\frac{1}{2}\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{T}{2}\right)^2 - (T-\theta)\; Y[T-\theta] \right] dx \tag{5}$$

As Eq. (1) represents a gradient system, F behaves as a Lyapunov functional.[7] and we can rewrite it as $dT(x, t)/dt = -\delta F\{T\}/\delta T$. During its temporal evolution, F decreases through the steepest descent trajectory until it reaches one of its minima. The unstable structures are related to extrema of F of the saddle-point type and define the magnitude of the barrier between the different locally stable attractors.[4, 16, 18]

III. CRITICAL EXPONENT

The nonequilibrium potential for gradient systems admits the following asymptotic perturbative expansion, in terms of t, around a stable stationary structure $T(x)$:[14]

$$F\{T(x) + \varepsilon(x, t)\} = F\{T(x)\} + (1/2)\,|C_0|^2 \exp(-2\lambda_0\, t) + \cdots \tag{6}$$

where

$$C_0 = \int_{-L}^{L} \varepsilon(x, t=0)\, \Phi_0(x)\, dx \tag{7}$$

and we discard faster exponentially decaying modes with amplitudes of order epsilon-square. $\Phi_0(x)$ is the normalized eigenfunction associated with the lowest eigenvalue $\lambda_0(L, \theta)$ of the linear stability analysis for the stable stationary structure. In the critical region, where marginal stability occurs, we expect a CSD in the dynamics of the system. That is also suggested by the time dependence of the nonequilibrium potential.

We have simulated the process of decay of perturbations near the critical region by finite differences. We have chosen patterns in the vicinity of the saddle point mentioned before (i.e., the unstable stationary structures slightly perturbed) as initial conditions.[17] In Fig. 1b we show the time evolution of F following small perturbations of unstable patterns, for several values of $\Delta\theta$. The decay to the beyond-threshold attractor results to be critically damped for $\Delta\theta \to 0$.

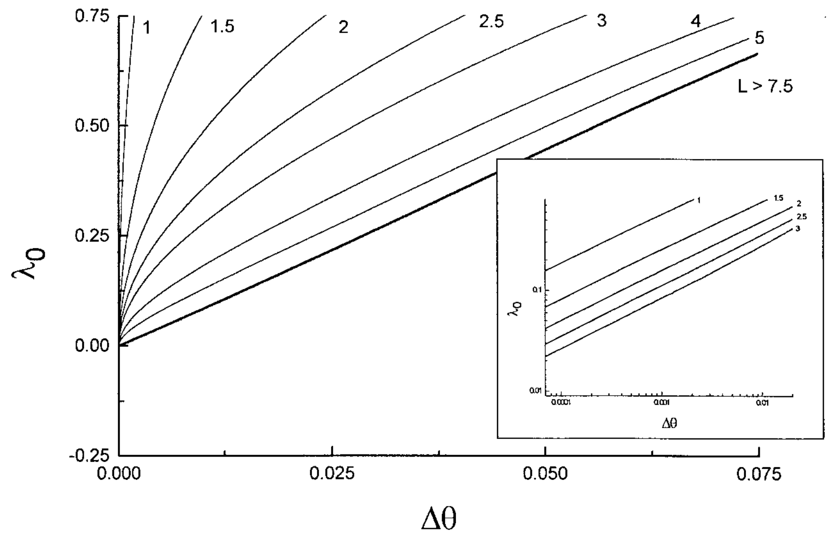

Fig. 2. Damped relaxation: λ_0 vs. $\Delta\theta$. The curves are labeled with the corresponding values of L. Marginal stability occurs for $\Delta\theta \to 0$. For $L \gtrsim 7.5$, where the size of the system no longer affects the value of θ_c, the curves coalesce over a straight line. **Insert:** λ_0 vs. $\Delta\theta$ in a log-log scale for small and moderate values of L. The curves are labeled with the corresponding values of L.

From Eq. (6) we can link the time-scale associated with the asymptotic evolution of the system, namely τ_R, to the lowest linear-stability eigenvalue of the stable attractor $\lambda_0 = 2\tau_R^{-1}$. We have followed the behavior of that eigenvalue in the neighborhood of the critical point. In Fig. 2 we show λ_0 vs. $\Delta\theta$ for several values of L. For $L \gtrsim 7.5$, where the size effect is no longer relevant to θ_c, the curves associated with different values of L coalesce and λ_0 scales as:

$$\lambda_0(L, \theta) \sim \Delta\theta \tag{8}$$

In that region, the relaxation time behaves therefore as $\tau_R \sim \Delta\theta^{-1}$. In Fig. 2 we see that the time scale of the relaxation changes when L decreases. In the insert of Fig. 2, we show λ_0 vs. $\Delta\theta$ in a log-log scale for selected values of L. For small and medium L (in fact for $4.5 \gtrsim L$), λ_0 scales as:

$$\lambda_0(L, \theta) \sim \sqrt{\Delta\theta} \tag{9}$$

Therefore, in that region, the relaxation time is given by $\tau_R \sim \Delta\theta^{-1/2}$. For any value of L, the extension of the activated region of the patterns in the environment of the critical point is always $2x_c = L \pm \sqrt{\Delta\theta}$.

IV. SUMMARY AND ASSESSMENT

We have studied a piecewise linear bistable reaction-diffusion system, which models superconducting microbridges, with the aim of investigating the nature of the decay in the critical region, where the system shows a structural instability.

The dynamical properties of the system were analyzed through its nonequilibrium potential. That kind of approach has been scarcely used for RD systems because the required potential conditions are not fulfilled in general. Several applications have been already developed for the case in which Eq. (1) is perturbed with random noise. In particular, the stationary probability distribution is directly related to $F\{T\}$. Barriers and escape times from metastable states in extended dissipative systems have been calculated as well.[16, 19–22]

We have shown here that in the vicinity of the critical point, the parameter $\Delta\theta(L)$ controls the extension of the activated region (i.e., the dissipation) and the damped relaxation time. The time evolution of $F\{T\}$ shows a critical slowing down in its march towards the stationary states; its time scale is given by the damped relaxation time τ_R, which depends on L and whose inverse is measured by the distance to the critical point in the space of the parameters.

For ballast resistor-like RD systems, the damped relaxation time can be tuned through $\Delta\theta$. We expect similar results in more elaborated bistable multicomponent models for which the implementation of the present scheme will offer an adequate framework. In particular, Eq. (6) can be straightforwardly generalized to multicomponent RD equations, provided they are gradient systems.

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REFERENCES

1. G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977).
2. H. Haken, *Synergetics: An Introduction*, 2nd. Ed. (Springer-Verlag, Berlin, 1978).
3. M. Cross and P. Hohenberg, *Rev. Mod. Phys.* **65**:851 (1993).
4. A. Mikhailov, *Foundations of Synergetics I* (Springer-Verlag, Berlin, 1990).
5. P. Fife, *Mathematical Aspects of Reacting and Diffusing Systems* (Springer-Verlag, Berlin, 1979).
6. H. Wio, *An Introduction to Stochastic Processes and Nonequilibrium Statistical Physics* (World Scientific, Singapore, 1994).
7. G. Izús, J. Reyes de Rueda, and C. Borzi, *J. Stat. Phys.* **90**:103 (1998).
8. D. Bedeaux, P. Mazur, and R. Pasmanter, *Physica A* **86**:355 (1977).
9. D. Bedeaux and P. Mazur, *Physica A* **105**:1 (1981).
10. W. Skocpol, M. Beasley, and M. Tinkham, *J. Appl. Phys.* **45**:4054 (1974).
11. C. Schat and H. Wio, *Physica A* **180**:295 (1992).
12. H. Frisch, V. Privman, C. Nicolis, and G. Nicolis, *J. Phys. A* **23**:1147 (1990).
13. V. Privman and H. Frisch, *J. Chem. Phys.* **94**:8216 (1991).
14. B. von Haeften, G. Izús, R. Deza, and C. Borzi, *Phys. Lett. A* **236**:403 (1997).
15. G. Izús, R. Deza, C. Borzi, and H. Wio, *Physica A* **237**:135 (1997).
16. D. Zanette, H. Wio, and R. Deza, *Phys. Rev. E* **53**:353 (1996).
17. F. Castelpoggi, H. Wio, and D. Zanette, *Int. J. Mod. Phys. B* **11**:1717 (1997).
18. R. Montagne, E. Hernandez-Garcia, and M. San Miguel, *Phys. D* **96**:47 (1996).
19. G. Izús, R. Deza, H. Wio, and C. Borzi, *Phys. Rev. E* **55**:4005 (1997).
20. G. Izús, H. Wio, J. Reyes de Rueda, O. Ramírez, and R. Deza, *Int. J. Mod. Phys. B* **10**:1273 (1996).
21. R. Graham, in *Quantum Statistics in Optics and Solid-State Physics* (G. Hólder, ed.), Springer Tracts in Modern Physics, Vol. 66, p. 1 (Springer-Verlag, Springer, 1973).
22. P. Hanggi, P. Talkner, and M. Borkovec, *Rev. Mod. Phys.* **62**(2): (1990).

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